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Unsteady Stokes' flow in two dimensions

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Abstract. Some simple problems are considered which indicate how transient effects lead to the development of slow viscous flows. Explicit solutions are obtained for situations when a source, rotlet or stokeslet is impulsively introduced at a particular time, and the manner by which such flows, which are initially harmonic, are transformed into those satisfying the biharmonic equation is clearly displayed. Conclusions regarding the formation of separated regions are presented.

1. Introduction

A Stokes' flow represents motion in a viscous fluid with low Reynolds number where the convection terms are taken to be negligibly small. Such behaviour is invariably considered to be independent of time, predominantly because if there is some variation over time it can be accommodated by taking the flow to be quasi-steady, with the external variable (the velocity of the body, for example) being the time-dependent parameter in the otherwise steady flow. Even then, there is difficulty in finding second-order effects, which would include derivatives of this external variable, and the recent article by Lawrence and Weinbaum [1] gives a good indication of the nature of the problems involved.

However, when there is an impulsive beginning to the motion from a state of rest, the small steady-state velocities are reached quickly, and so the development of Stokes' flows would seem to have limited relevance; certainly, there have been few solutions presented to the time-dependent linearized Navier–Stokes equations, and most of these are concerned with Rayleigh flow (the impulsively started motion of a plane parallel to its generator) and its generalizations – c.f. Rosenhead [2, p. 137]. Nevertheless, there is value in understanding how diffusion can act to bring about steady viscous motions, particularly in describing the transition from potential to Stokes' flows, and this is the focus of the present study. To be precise, it is the motions where diffusion alone brings about the final steady state which we designate as unsteady Stokes' flows.

In two dimensions, with co-ordinates (X, Y), the Navier–Stokes equations at time T are

$$\Omega_T - \frac{\partial(\Psi, \Omega)}{\partial(X, Y)} = \nu(\Omega_{XX} + \Omega_{YY}), \quad \Omega = \Psi_{XX} + \Psi_{YY}, \tag{1.1}$$

for coefficient of viscosity ν, where Ψ is the stream function and Ω the vorticity. If a is the length scale, and U the velocity scale for any particular motion, then writing $X = ax$, $Y = ay$, $\Psi = Ua\psi$, $\Omega = Ua^{-1}\omega$, plus $T = a^2\nu^{-1}t$, leads to the non-dimensional form of

(1.1) as

$$\omega_t - R\frac{\partial(\psi, \omega)}{\partial(x, y)} = \omega_{xx} + \omega_{yy}, \quad \omega = \psi_{xx} + \psi_{yy}, \tag{1.2}$$

where R is the Reynolds number given by $R = Ua\nu^{-1}$, which we are assuming to be small. There are clear implications of the scaling adopted. Firstly, we have taken a as the length scale rather than the smaller viscous length ν/U, but because all the motions described here are created by a singularity (source, rotlet or stokeslet), the local behaviour in the neighbourhood (i.e., within a distance ν/U) of the singularity, is of limited interest. Secondly, the time is scaled by a^2/ν, to ensure that the diffusion quantities $X^2/\nu T$ and $Y^2/\nu T$ are the O(1) variables x^2/t and y^2/t throughout the development of the flow; clearly, finite changes in t represent rapid changes in T. Consequently, in setting $R \ll 1$, diffusion is the only action included to bring about a steady Stokes' flow after the impulsive start.

One major complication in considering two-dimensional flows is a consequence of the Stokes' paradox, and this question is addressed in Section 2. To illustrate, for the flow due to an impulsively started circular cylinder the behaviour at infinity cannot be prescribed for all time (initially it would be at rest, but finally it must correspond to a stokeslet), and so the formulation of the problem to describe this motion is incomplete. However, this difficulty is avoided in the following by designating the strength of the singularity for all time, though a consequence is that the velocities can become unbounded for large t.

In Section 3, the development of the viscous flow due to a source of fluid placed in a plane wall is presented, and the corresponding behaviour due to a point stress along a wall, where there is no initial potential flow, is sketched for comparison in Section 4. Next, we consider the development of the flow due to a rotlet in front of a plane wall, and find that it takes a finite time before the separation of the flow from the wall develops. The results for a stokeslet in front of a wall are given in Section 6; the conclusion of note is that when the stokeslet is parallel to the wall then a separated region immediately forms between the singularity and the wall, but collapses as the time increases and is completely absent in the steady-state situation. There has been considerable interest recently concerning the existence of separation for steady Stokes' flows; here it is seen that there can be separation in the transient stage, even when it is not present for the ultimate steady-state flow.

In only investigating behaviours described by a singularity the analysis is thereby simplified, and does permit a solution for the velocities to be given in terms of standard functions no more complex than the error function, which are readily interpreted. The mathematics involved is rather lengthy, though essentially just straightforward transform analysis; because the main purpose here is to observe the manner of the transient development, the solutions are presented with limited detail.

2. Singular solutions in an infinite fluid

The velocities in the x and y directions are written as $u(x, y, t)$ and $v(x, y, t)$; $p(x, y, t)$ is the pressure. If a stokeslet, which represents a unit point force in the x-direction, is created impulsively at the origin at time $t = 0$, then the equations are

$$u_x + v_y = 0, \tag{2.1}$$

$$u_t = -p_x + u_{xx} + u_{yy} + \delta(x)\delta(y)H(t), \tag{2.2}$$

$$v_t = -p_y + v_{xx} + v_{yy}; \tag{2.3}$$

all the variables are scaled to make them non-dimensional, as with (1.2). When the pressure function is eliminated,

$$\omega_t = \nabla^2\omega + \delta(x)\delta'(y)H(t), \quad \omega = \nabla^2\psi, \tag{2.4}$$

with ψ given by $u = \psi_y, v = -\psi_x$. Defining Fourier transforms in x and y, plus a Laplace transform in t, by

$$\bar{\psi}(\alpha, \beta, s) = \int_{-\infty}^{\infty} e^{-i\beta y}\, dy \int_{-\infty}^{\infty} e^{-i\alpha x}\, dx \int_0^{\infty} e^{-st}\psi(x, y, t)\, dt,$$

with an equivalent definition for $\bar{\omega}$, then the equations (2.4) can be transformed to an algebraic equation for $\bar{\psi}$ which shows

$$\bar{\psi} = -\frac{i\beta}{s^2}\left\{\frac{1}{\alpha^2 + \beta^2} - \frac{1}{\alpha^2 + \beta^2 + s}\right\}. \tag{2.5}$$

When the inverse Fourier transforms are taken, followed by the inverse Laplace transform, it can be seen from standard results (c.f. Erdelyi et al., [3]) that

$$\psi = \frac{y}{8\pi}\left\{\frac{4t}{r^2}(1 - e^{-r^2/(4t)}) - \mathrm{Ei}\left(-\frac{r^2}{4t}\right)\right\}, \quad r^2 = x^2 + y^2, \tag{2.6}$$

where $\mathrm{Ei}(-z) = -\int_z^{\infty} v^{-1} \cdot e^{-v}\, dv$ is the exponential integral; the vorticity is given by

$$\omega = -\frac{y}{2\pi r^2} e^{-r^2/(4t)}.$$

From (2.6) it can be seen that

$$\psi = \frac{yt}{2\pi r^2} + O\left\{tr^{-3}e^{-r^2/(4t)}\right\}, \quad 0 < t \ll 1,$$

which indicates a dipole whose strength is growing linearly in t. However, for large t, it follows that ψ grows without bound as $(8\pi)^{-1}y \ln t$, which represents a uniform stream of increasing velocity. The exponential integral $\mathrm{Ei}(-z)$ behaves as $\ln z$ for small positive z, and so, because diffusion is the dominant process, acting through the similarity variable r^2/t, the singularity for large time follows necessarily from the singularity for large distances which is present in two-dimensional Stokes' flows – the Stokes' paradox.

Next, if a rotlet, which represents a unit torque, is created impulsively at the origin at time $t = 0$, then the corresponding equations to (2.4) are

$$\omega_t = \nabla^2\omega - \{\delta(x)\delta''(y) + \delta''(x)\delta(y)\}H(t), \quad \omega = \nabla^2\psi; \tag{2.7}$$

these are the cartesian co-ordinate equivalent of the equation $V_t = V_{rr} + r^{-1}V_r - r^{-2}V + r^{-1}\delta'(r)H(t)$ for the azimuthal velocity $V(r, t)$ in polar co-ordinates. The solution of (2.7), gained by a similar approach to that for (2.4), shows $\bar{\psi} = -s^{-1}(\alpha^2 + \beta^2 + s)^{-1}$, so that

$$\psi = \frac{1}{4\pi}\,\mathrm{Ei}\left(-\frac{r^2}{4t}\right), \quad \omega = -\frac{1}{4\pi t}\,\mathrm{e}^{-r^2/(4t)};$$

this has the same singular behaviour for $t \to \infty$ as noted for the stokeslet.

It is clear that this factor puts a definite restriction on the value of unsteady two-dimensional Stokes' flows which, fortunately, is not a difficulty at all in three dimensions. However, in all but one of the problems considered in the following sections, where we are interested in the interaction of a singularity with a plane wall, the image system includes an equal and opposite singularity at the point of reflection. Consequently, the unbounded components cancel, and a bounded solution is possible both for large distances from the singularity and for large times.

3. Source of fluid in a plane wall

A source of fluid is placed at the origin in the wall $y = 0$, and commences to discharge into the upper half-plane $y > 0$ at $t = 0$. The equations to be satisfied are $\omega_t = \nabla^2\omega$, $\omega = \nabla^2\psi$, subject to the boundary conditions

$$\psi = \operatorname{sgn} x H(t), \quad \psi_y = 0 \quad \text{on} \quad y = 0; \tag{3.1}$$

the finite jump $[\psi(x, 0, t)]_{x=0-}^{x=0+} = 2H(t)$ represents the mass flux from the source. As $t \to 0+$, the flow is potential with stream function

$$\psi_0 = \psi(x, y, 0) = 1 - \frac{2}{\pi}\arctan\frac{y}{x} = \frac{2\phi}{\pi}$$

in terms of a polar co-ordinate system defined by $x = r\sin\phi$, $y = r\cos\phi$; the steady-state Stokes' flow is represented by

$$\psi_\infty = \psi(x, y, \infty) = 1 - \frac{2}{\pi}\arctan\frac{y}{x} + \frac{2}{\pi}\frac{xy}{x^2 + y^2} = \frac{2\phi}{\pi} + \frac{1}{\pi}\sin 2\phi.$$

We wish to find the transition from the one flow to the other. It is noted immediately that ψ is odd in x, and so it is sufficient to take $x, y, t \geqslant 0$.

The approach is to write $\psi = \psi_0(x, y) + \psi_1(x, y, t)$, thereby assuring that ψ_1 is continuous throughout the fluid. Then, when Laplace transforms in t, and Fourier sine transforms in x are taken, through

$$\tilde{\psi}(\alpha, y, s) = \int_0^\infty e^{-st}\,dt \int_0^\infty \sin \alpha x \cdot \psi(x, y, t)\,dx, \tag{3.2}$$

it follows that

$$\tilde{\omega}_{1yy} = (\alpha^2 + s)\tilde{\omega}_1, \quad \tilde{\omega}_1 = \tilde{\psi}_{1yy} - \alpha^2\tilde{\psi}_1, \tag{3.3}$$

so that the general expression for $\tilde{\psi}$ is

$$\tilde{\psi}(\alpha, y, s) = A(\alpha, s)e^{-\alpha y} + B(\alpha, s)e^{-(\alpha^2+s)^{1/2}y} + \frac{e^{-\alpha y}}{\alpha s}, \tag{3.4}$$

with the third term in (3.4) representing the transform of ψ_0. The boundary condition (3.1) for ψ requires $A + B = 0$, and for ψ_y requires $\alpha A + (\alpha^2 + s)^{1/2}B + s^{-1} = 0$, to give

$$\tilde{\psi} = \frac{e^{-\alpha y}}{s}\left\{\frac{1}{\alpha} + \frac{1}{(\alpha^2+s)^{1/2} - \alpha}\right\} - \frac{e^{-(\alpha^2+s)^{1/2}y}}{s\{(\alpha^2+s)^{1/2} - \alpha\}}. \tag{3.5}$$

The formal solution is complete with the inverse transform

$$\psi(x, y, t) = \frac{1}{\pi^2 i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s}\,ds \int_0^\infty \sin \alpha x \left\{\frac{e^{-\alpha y}}{\alpha} + \frac{e^{-\alpha y} - e^{-(\alpha^2+s)^{1/2}y}}{(\alpha^2+s)^{1/2} - \alpha}\right\} d\alpha. \tag{3.6}$$

Before considering the evaluation of (3.6) we see that the vorticity $\omega(x, y, t)$ is given by the more simple expression

$$\omega(x, y, t) = -\frac{1}{\pi^2 i}\int_{c-i\infty}^{c+i\infty} e^{st}\,ds \int_0^\infty \frac{\sin \alpha x \cdot e^{-(\alpha^2+s)^{1/2}y}}{(\alpha^2+s)^{1/2} - \alpha}\,d\alpha. \tag{3.7}$$

The inverse Laplace transform (c.f. Erdelyi et al. [3]), plus some simplification, leads to

$$\omega = \frac{i}{\pi t}e^{-r^2/(4t)}\,\mathrm{erf}\left(\frac{ix}{2t^{1/2}}\right) - \frac{2}{\pi^{1/2}}\int_0^\infty \alpha e^{-\alpha y} \sin \alpha x \cdot \mathrm{erfc}\left(\frac{y}{2t^{1/2}} - t^{1/2}\right) d\alpha. \tag{3.8}$$

The remaining integral in (3.8) is typical of those we find throughout the computations in this paper. To evaluate it we first integrate by parts (through knowing the indefinite integral with respect to α of $\alpha e^{-\alpha y} \sin \alpha x$), thereby replacing the complementary error function by an exponential, and then using results from Erdelyi et al. to deduce the resulting integrals which are of the form $\int_0^\infty \alpha^n e^{i\alpha x - \alpha^2 t} d\alpha$ for integers n. The details are lengthy, but straightforward, and finally show

$$\omega = -\frac{1}{2\pi t}\Bigg[\left(1+\frac{4}{\varrho^2}\right)e^{-\varrho^2/4}\sin 2\phi + ie^{-\varrho^2/4}\,\mathrm{erf}\left(\frac{i\eta}{2}\right)$$

$$+\left\{\frac{4\sin\phi}{\varrho}\left(\frac{1}{\pi^{1/2}}e^{-\zeta^2/4} - \frac{2\cos\phi}{\varrho}\,\mathrm{erfc}\,(\tfrac{1}{2}\zeta)\right)\right.$$

$$\left.- i\left(1+\frac{4}{\varrho^2}\right)e^{-\varrho^2/4}\,\mathrm{erf}\left(\frac{i\eta}{2}\right)\cos 2\phi\right\}\Bigg], \tag{3.9}$$

where $\varrho = r/t^{1/2}$, $\eta = x/t^{1/2}$, $\zeta = y/t^{1/2}$. The (real) function

$$F(\eta) = -ie^{-\eta^2/4}\,\mathrm{erf}\left(\frac{i\eta}{2}\right) = \frac{2}{\pi^{1/2}}\int_0^{\eta/2} e^{-u(\eta-u)}\cdot \mathrm{d}u \tag{3.10}$$

is algebraic in nature with

$$F(\eta) = \pi^{-1/2}\eta\,(1 - \tfrac{1}{6}\eta^2) + O(\eta^5) \quad \text{as} \quad \eta \to 0,$$

and

$$F(\eta) = \frac{2}{\pi^{1/2}\eta}\left(1+\frac{2}{\eta^2}\right) + O(\eta^{-5}) \quad \text{as} \quad \eta \to \infty;$$

the graph is given in Fig. 1.

It is observed from (3.9) that, because there is no length scale in the problem as posed, it is possible to write $\omega = t^{-1}H(\eta, \zeta)$, for some function H which, when substituted into the

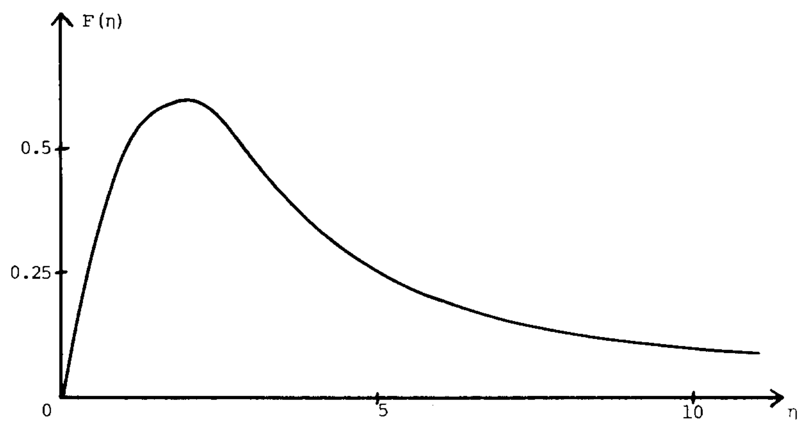

Fig. 1. Graph of the function $F(\eta)$ as defined by (3.10).

vorticity equation, satisfies

$$H_{\eta\eta} + H_{\zeta\zeta} + \tfrac{1}{2}\eta H_\eta + \tfrac{1}{2}\zeta H_\zeta + H = 0, \tag{3.11}$$

or, its equivalent form,

$$H_{\varrho\varrho} + \tfrac{1}{2}\varrho H_\varrho + \varrho^{-1} H_\varrho + \varrho^{-2} H_{\phi\phi} + H = 0. \tag{3.12}$$

Each of the three terms in (3.9) separately satisfy the equation for H, and the solution can be seen as the particular linear combination which satisfies the boundary conditions on $y = 0$. It is clear that the first term of (3.9) represents (focused) diffusion from the origin only and, being proportional to sin 2ϕ, can be identified as an eigenfunction for a flow with zero vorticity on the wall; the other two terms have exponential behaviour in ζ only, and so are more representative of diffusion from the wall itself.

Different approximations to (3.9) can be taken for small and large t to gain an understanding as to now the transition from potential to Stokes' flow takes place. For example, as $t \to 0$,

$$\omega \simeq -\frac{\sin 2\phi \cdot e^{-\varrho^2/4}}{2\pi t} - \frac{4}{\pi^{3/2} t^{1/2}} \frac{\sin \phi}{r} e^{-\zeta^2/4} - \frac{2 \sin 2\phi}{\pi r^2} \left\{ e^{-\varrho^2/4} + \operatorname{erfc}\left(\frac{\zeta}{2}\right) \right\}. \tag{3.13}$$

The first term is focused at the origin, and the second represents the Rayleigh-type behaviour in the developing shear layer along $y = 0$, while the third is a combination of both effects; it is clear that the first term could not be given by a boundary-layer approach, yet it is the leading term at finite r as $t \to 0+$ for all unsteady viscous flows – independent of the Reynolds number. For large t, the decay to the steady-state behaviour is a quite rapid algebraic decay with

$$\omega \simeq -\frac{4 \sin 2\phi}{\pi r^2} - \frac{r \sin \phi}{3(\pi t)^{3/2}} + \frac{r^2 \sin 2\phi}{16\pi t^2} + \frac{r^3(15 \sin \phi + 7 \sin 3\phi)}{240\pi^{3/2} t^{5/2}}. \tag{3.14}$$

The first three terms are harmonic in r, ϕ, and there are, in fact, an infinite set of harmonic terms of the form $r^n \sin n\phi \cdot t^{-1-n/2}$; there is also a second infinite set of the form $r^{n+2} \sin n\phi \cdot t^{-2-n/2}$ (the first of the set being present in the fourth term of (3.14)) which is the

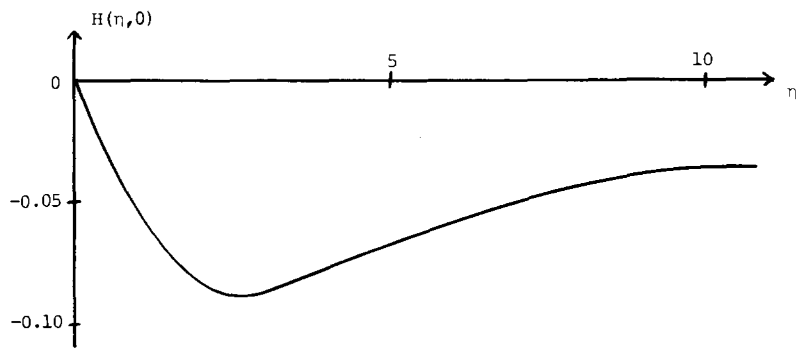

Fig. 2. Vorticity on the wall for the flow out of a source placed at the origin: $\omega(x, 0, t) = t^{-1}G(\eta)$ as given in (3.15).

response to the first set through the vorticity equation. Finally, on the plane $y = 0$ we have

$$\omega(x, 0, t) = -\frac{2}{\pi\eta^2 t}\{\pi^{-1/2}\eta - F(\eta)\}, \tag{3.15}$$

which is negative for all η, and tends to zero as $\eta \to 0$ (i.e., as $t \to \infty$); c.f. Fig. 2. It is one of the special result for Stokes' behaviour for flows along a plane wall that the vorticity is zero on the wall, yet we see that this is only the situation in the limiting steady-state case.

The inverse transform (3.6) for the stream function cannot be completely given in terms of standard functions because of one particular integral; however, the velocities can be so given, though after many pages of detail, and we just record them here by

$$u = \frac{2}{\pi t^{1/2}}\left[-\frac{\sin\phi}{\varrho} - \frac{2\sin 3\phi}{\varrho^3} + \frac{2\sin 2\phi}{\pi^{1/2}\varrho^2}(1 + e^{-\zeta^2/4})\right.$$

$$- \left\{\frac{\sin 3\phi - 3\sin\phi}{4\varrho} - \frac{2\sin 3\phi}{\varrho^3}\right\}\operatorname{erfc}(\tfrac{1}{2}\zeta) + \left\{\frac{\sin 3\phi + \sin\phi}{4\varrho} + \frac{2\sin 3\phi}{\varrho^3}\right\}$$

$$\times\, e^{-\varrho^2/4} + i\left\{\frac{\cos 3\phi + 3\cos\phi}{4\varrho} + \frac{2\cos 3\phi}{\varrho^3}\right\} e^{-\varrho^2/4}\operatorname{erf}(\tfrac{1}{2}i\eta)$$

$$\left. - \left\{\frac{i}{(\zeta + i\eta)^3} e^{(\zeta + i\eta)^2/4}\operatorname{erfc}[\tfrac{1}{2}(\zeta + i\eta)] - \frac{i}{(\zeta - i\eta)^3} e^{(\zeta - i\eta)^2/4}\operatorname{erfc}[\tfrac{1}{2}(\zeta - i\eta)]\right\}\right],$$

$$v = \frac{2}{\pi t^{1/2}}\left[-\frac{\cos\phi}{\varrho} - \frac{2\cos 3\phi}{\varrho^3} - \frac{2\cos 2\phi}{\pi^{1/2}\varrho^2}(1 - e^{-\zeta^2/4})\right.$$

$$- \left\{\frac{\cos 3\phi - \cos\phi}{4\varrho} - \frac{2\cos 3\phi}{\varrho^3}\right\}\operatorname{erfc}(\tfrac{1}{2}\zeta) + \left\{\frac{\cos 3\phi - \cos\phi}{4\varrho} + \frac{2\cos 3\phi}{\varrho^3}\right\}$$

$$\times\, e^{-\varrho^2/4} + i\left\{\frac{\sin 3\phi + \sin\phi}{4} + \frac{2\sin 3\phi}{\varrho^3}\right\} e^{-\varrho^2/4}\operatorname{erf}(\tfrac{1}{2}i\eta)$$

$$\left. - \left\{\frac{1}{(\zeta + i\eta)^3} e^{(\zeta + i\eta)^2/4}\operatorname{erfc}[\tfrac{1}{2}(\zeta + i\eta)] + \frac{1}{(\zeta - i\eta)^3} e^{(\zeta - i\eta)^2/4}\operatorname{erfc}[\tfrac{1}{2}(\zeta - i\eta)]\right\}\right].$$

Having to take the sum of complementary error functions with complex arguments is a little awkward, and does somewhat limit the benefit of having explicit solutions; however, it can be seen that the (potential function) combination

$$G(\eta, \zeta) = e^{(\zeta + i\eta)^2/4}\operatorname{erfc}[\tfrac{1}{2}(\zeta + i\eta)] + e^{(\zeta - i\eta)^2/4}\operatorname{erfc}[\tfrac{1}{2}(\zeta - i\eta)]$$

is essentially algebraic with

$$G = 2 - 2\pi^{-1/2}\varrho\cos\phi + \tfrac{1}{2}\varrho^2\cos 2\phi - \tfrac{1}{3}\pi^{-1/2}\varrho^3\cos 3\phi + O(\varrho^4), \quad \varrho \text{ small},$$

$$G = \pi^{-1/2}\{4\varrho^{-1}\cos\phi - 8\varrho^{-3}\cos 3\phi + 48\varrho^{-5}\cos 5\phi + O(\varrho^{-7})\}, \quad \varrho \text{ large}.$$

For small time t, approximations to (3.6) show

$$\psi \simeq \frac{2}{\pi}\left\{\phi + \frac{4 \sin \phi}{\pi^{1/2}\varrho} + \frac{\sin 2\phi}{\varrho^2} + \frac{8 \sin 3\phi}{\pi^{1/2}\varrho^3}\right\}, \quad \varrho \gg 1;$$

there is the set of harmonic terms $O(t^{n/2}r^{-n} \sin n\phi)$ representing the multipoles for the potential part of the flow, where the lower orders of multipoles grow the most quickly in time. There are also the fully viscous terms which could be derived from (3.13), which have an essential singularity at $t = 0$.

4. Point stress on a plane wall

A corresponding solution can be found when we consider the flow in the half-space $y \geqslant 0$ described by $\omega_t = \nabla^2\omega$, $\omega = \nabla^2\psi$, subject to

$$\psi = 0, \quad \psi_y = \delta(x)H(t) \text{ on } y = 0. \tag{4.1}$$

These boundary conditions for the steady flow can be formulated as follows: the plane $y = 0$ has a segment $|x| < a$ which slides with constant velocity U underneath the rest of the plane $|x| > a$ which is at rest, thereby requiring $\psi = 0$ and $\psi_y = U\{H(x - a) - H(x + a)\}$ on $y = 0$; when $a \to 0$ and $U \to \infty$ such that $2Ua \to 1$, the condition (4.1) results. The motion of the fluid is created solely by viscous forces, and the steady-state behaviour can be seen to be given by $\psi = \pi^{-1}y^2/(x^2 + y^2)$.

The solution of the problem as formulted is gained by equivalent methods to those of Section 3, and we quote the results as

$$\psi(x, y, t) = \frac{1}{2\pi^2 i}\int_{c - i\infty}^{c + i\infty} \frac{e^{st}}{s}\, ds \int_0^\infty \cos \alpha x \cdot \frac{e^{-\alpha y} - e^{-(\alpha^2 + s)^{1/2}y}}{(\alpha^2 + s)^{1/2} - \alpha}\, d\alpha, \tag{4.2}$$

which is the corresponding even (in x) part of the odd (in x) expression given in (3.6); the vorticity which follows from (4.2) is

$$\omega(x, y, t) = -\frac{1}{2\pi^2 i}\int_{c - i\infty}^{c + i\infty} e^{st}\, ds \int_0^\infty \cos \alpha x \cdot \frac{e^{-(\alpha^2 + s)^{1/2}y}}{(\alpha^2 + s)^{1/2} - \alpha}\, d\alpha, \tag{4.3}$$

which can be evaluated for

$$\omega = -\frac{1}{2\pi t}\left[e^{-\varrho^2/4} + \frac{1}{2}\left(1 + \frac{4}{\varrho^2}\right)\cos 2\phi \cdot e^{-\varrho^2/4}\right.$$
$$\left. + \left\{\frac{2 \cos \phi}{\pi^{1/2}\varrho}\, e^{-\zeta^2/4} + \frac{2 \cos 2\phi}{\varrho^2}\operatorname{erfc}\left(\frac{\zeta}{2}\right) + \frac{i}{2}\left(1 + \frac{4}{\varrho^2}\right)e^{-\varrho^2/4}\operatorname{erf}\left(\frac{i\eta}{2}\right)\sin 2\phi\right\}\right]. \tag{4.4}$$

We note, as before, that the square brackets represent the sum of three distinct solutions of the equation (3.11), though now both the first and second represent diffusion focused at the origin. For small time t,

$$\omega \simeq -\frac{\sin^2\phi}{2\pi t}\,\mathrm{e}^{-\varrho^2/4} - \frac{1}{2\pi^{3/2}t^{1/2}}\frac{\cos\phi}{r}\,\mathrm{e}^{-\zeta^2/4},$$

and for large time,

$$\omega \simeq \frac{\cos 2\phi}{\pi r^2} - \frac{1}{4\pi t} + \frac{r\cos\phi}{12(\pi t)^{3/2}} + \frac{r^2\cos 2\phi}{32\pi t^2} - \frac{r^2}{16\pi t^2};$$

similar interpretations for these expressions to those given in the previous section are valid here. Finally, the velocities can be found in terms of standard functions, and although not given here we do note that for small time $\psi \simeq 2\pi^{-3/2}t^{1/2}r^{-1}\cos\phi$, which represents a dipole with a strength which grows as $t^{1/2}$; the higher-order multipoles grow more slowly.

5. Rotlet in front of a plane wall

We next consider the flow due to a point rotlet at (0, 1) in fluid which occupies the region $y \geqslant 0$. When there is no wall, the solution was described in Section 2. The steady-state behaviour with the wall was given by Ranger [4], and represents one of the simplest Stokes' flows which leads to separation. The rotlet is turned on at $t = 0$, so as $t \to 0+$ the stream function will be the potential flow

$$\psi_0 = \tfrac{1}{2}\ln\frac{x^2 + (y-1)^2}{x^2 + (y+1)^2}. \tag{5.1}$$

It is therefore necessary to solve $\omega_t = \nabla^2\omega$, $\omega = \nabla^2\psi$ for all $x, y, t \geqslant 0$ (the solution will be even in x), where $\psi = \psi_0$ at $t = 0$ and $\psi = \psi_y = 0$ on $y = 0$. The transform is defined (with slight variation from Section 3) by

$$\tilde{\psi}(\alpha, y, s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t \int_0^\infty \psi(x, y, t)\cos\alpha x\,\mathrm{d}\alpha,$$

which leads to

$$\tilde{\psi} = \frac{\pi}{2\alpha s}\{\mathrm{e}^{-\alpha(y+1)} - \mathrm{e}^{-\alpha|y-1|}\} + \frac{\pi\mathrm{e}^{-\alpha}}{s}\cdot\frac{\mathrm{e}^{-\alpha y} - \mathrm{e}^{-(\alpha^2+s)^{1/2}y}}{(\alpha^2+s)^{1/2} - \alpha}, \tag{5.2}$$

with

$$\tilde{\omega} = -\frac{\pi\mathrm{e}^{-\alpha-(\alpha^2+s)^{1/2}y}}{(\alpha^2+s)^{1/2} - \alpha}. \tag{5.3}$$

We can completely evaluate the inverse transform for $\tilde{\omega}$ by utilising results similar to those of Section 3 to provide, after some effort,

$$\begin{aligned}\omega(x, y, t) \;=\; & -\,e^{-y^2/(4t)}\left[\frac{y}{2t(y+1+ix)}+\frac{1}{(y+1+ix)^2}\right]e^{(1+ix)^2/(4t)}\,\mathrm{erfc}\left(\frac{1+ix}{2t^{1/2}}\right)\\ & -\,e^{-y^2/(4t)}\left[\frac{1}{2t(y+1-ix)}+\frac{1}{(y+1-ix)^2}\right]e^{(1-ix)^2/(4t)}\,\mathrm{erfc}\left(\frac{1-ix}{2t^{1/2}}\right)\\ & -\frac{2\{(y+1)^2-x^2\}}{\{(y+1)^2+x^2\}^2}\,\mathrm{erfc}\left(\frac{y}{2t^{1/2}}\right)-\frac{2(y+1)}{(\pi t)^{1/2}\{(y+1)^2+x^2\}}\,e^{-y^2/(4t)}.\end{aligned} \tag{5.4}$$

We limit our interpretation of the result (5.4) to the development of separation. Both steady and unsteady flows become detached, or separated, from a stationary wall at a point with zero skin friction (c.f. Telionis [5]); such are represented in the present context by points on the wall where $\omega(x, 0, t) = 0$.

As could have been expected, separation takes place first for large x; however, with the approximation to (5.4) for $x \gg 1$ indicating $\omega(x, 0, t) \simeq 2x^{-2}\{1 - (\pi t)^{-1/2}\}$, it is seen that it takes the finite time $t = \pi^{-1}$ before the separation commences, which is an unexpected conclusion. When the approximation is pursued further, and the time of separation at position x is written as $t_s(x)$, then

$$t_s(x) \;=\; \frac{1}{\pi}\left\{1+\left(4-\frac{8}{\pi}\right)x^{-2}+O(x^{-4})\right\},\quad x \gg 1,$$

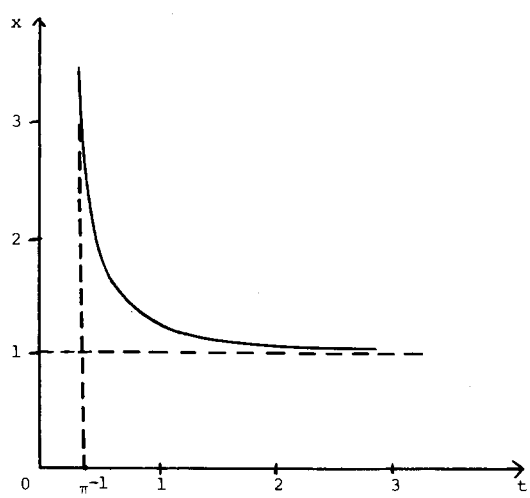

Fig. 3. Time of separation $t_s(x)$ for a point on the wall for the flow due to a rotlet.

showing how the point of separation commences to move inward towards its final position at $x = 1$, where a separate computation indicates

$$t_s(x) = \frac{1}{4(x-1)} - \frac{1}{3\pi^{1/2}(x-1)^{1/2}} + \left(\frac{4}{9\pi} + \frac{3}{8}\right) + O\{(x-1)^{1/2}\}$$

where $0 < x - 1 \ll 1$. The movement is shown graphically in Fig. 3; x is within 10% of its final value by $t_s = 3$.

6. Stokeslet in front of a plane wall

When the rotlet is replaced by a stokeslet at (0, 1) which represents a unit force in the x-direction, which commences to act at time $t = 0$, then a similar argument to that of the previous section shows that the vorticity $\omega(x, y, t)$ is given by

$$\omega = \frac{2}{\pi^2 i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s}\,ds \int_0^\infty \cos\alpha x \left\{ e^{-(\alpha^2+s)^{1/2}|y-1|} + \frac{\alpha[e^{-(\alpha^2+s)^{1/2}} - e^{-\alpha}]}{(\alpha^2+s)^{1/2} - \alpha}\, e^{-(\alpha^2+s)^{1/2}y} \right\} d\alpha. \quad (6.1)$$

We note that in this particular case the image system includes a stokeslet at the point of reflection (0, − 1) which is in the same direction as that at (0, 1); hence the non-uniformities for the stream function for large r and t are still present, although this does not affect the main conclusion developed here. The evaluation of the double integral (6.1) proceeds with similar, though even more extensive, calculations to the final result

$$\begin{aligned}
\pi\omega = {} & -\frac{2(y-1)}{x^2+(y-1)^2}\, e^{-[x^2+(y-1)^2]/(4t)} \\
& + \left[-\frac{2(y+1)}{x^2+(y+1)^2} + \frac{4(y+1)^3}{\{x^2+(y+1)^2\}^2} + \frac{8t(y+1)\{(y+1)^2-3x^2\}}{\{x^2+(y+1)^2\}^3} \right] \\
& \times e^{-[x^2+(y+1)^2]/(4t)} + \left[-\frac{y-1}{x^2+(y-1)^2} - \frac{y+1}{x^2+(y+1)^2} \right. \\
& \left. + \frac{2y\{(y+1)^2-x^2\}}{\{x^2+(y+1)^2\}^2} - \frac{8t(y+1)\{(y+1)^2-3x^2\}}{\{x^2+(y+1)^2\}^3} \right] \operatorname{erfc}\left(\frac{y}{2t^{1/2}}\right) \\
& - 8\left(\frac{t}{\pi}\right)^{1/2} \frac{(y+1)^2-x^2}{\{x^2+(y+1)^2\}^2}\, [e^{-y^2/(4t)} - e^{-(y+1)^2/(4t)}] \\
& + \left[\frac{4x^2(y+1)}{\{x^2+(y+1)^2\}^2} + \frac{8t(y+1)\{(y+1)^2-3x^2\}}{\{x^2+(y+1)^2\}^3} \right] \operatorname{erfc}\left(\frac{y+1}{2t^{1/2}}\right) \\
& + i\left[\frac{4x(y+1)^2}{\{x^2+(y+1)^2\}^2} + \frac{8tx\{3(y+1)^2-x^2\}}{\{x^2+(y+1)^2\}^3} \right] e^{-[x^2+(y+1)^2]/(4t)}
\end{aligned}$$

$$\times \operatorname{erfc}\left(\frac{ix}{2t^{1/2}}\right) + \left\{\left[\frac{1}{2}\frac{(y-1)-ix}{x^2+(y-1)^2} - \frac{3}{2}\frac{(y+1)+ix}{x^2+(y+1)^2}\right.\right.$$

$$+\frac{(y+2)\{(y+1)^2-x^2\}+4x^2(y+1)+2ix(x^2+1+y)}{\{x^2+(y+1)^2\}^2}$$

$$\left. -\frac{4t[(y+1)\{(y+1)^2-3x^2\}+ix\{3(y+1)^2-x^2\}]}{\{x^2+(y+1)^2\}^3}\right]$$

$$\left. \times\, e^{-[y^2-(1-ix)^2]/(4t)} \operatorname{erfc}\left(\frac{1-ix}{2t^{1/2}}\right) + [\text{complex conjugate}]\right\}. \qquad (6.2)$$

The vorticity on the plane wall $y = 0$ is given from (6.2) by the simpler formula

$$\pi\omega(x, 0, t) = -\frac{8t(1-3x)^2}{(1+x^2)^3} - 8\left(\frac{t}{\pi}\right)^{1/2}\frac{1-x^2}{(1+x^2)^2}(1-e^{-1/(4t)})$$

$$+\left[\frac{4x^2}{(1+x^2)^2}+\frac{8t(1-3x^2)}{(1+x^2)^3}\right]\operatorname{erfc}\left(\frac{1}{2t^{1/2}}\right)$$

$$+\left[\frac{4}{(1+x^2)^2}+\frac{8t(1-3x^2)}{(1+x^2)^3}\right]e^{-(1+x^2)/(4t)}$$

$$+\,i\left[\frac{4x}{(1+x^2)^2}+\frac{8tx(3-x^2)}{(1+x^2)^3}\right]e^{-(1+x^2)/(4t)}\operatorname{erf}\left(\frac{ix}{2t^{1/2}}\right)$$

$$-\left[\frac{4t\{(1-3x^2)+i(3-x^2)\}}{(1+x^2)^3}e^{(1-ix)^2/(4t)}\operatorname{erfc}\left(\frac{1-ix}{2t^{1/2}}\right)\right.$$

$$\left. + (\text{complex conjugate})\right]. \qquad (6.3)$$

This expression (6.3) is of a form which can be readily analysed, and the following conclusions emerge. Firstly, for small t, it is seen that

$$\pi\omega(x, 0, t) = -8\left(\frac{t}{\pi}\right)^{1/2}\frac{1-x^2}{(1+x^2)^2} - 8t\frac{1-3x^2}{(1+x^2)^3} + O(t^{3/2}) \qquad (6.4)$$

for finite x. Initially, the stokeslet at (0, 1) behaves as a dipole whose strength is proportional to t (as seen in Section 2), and so the image system has a dipole of equal strength at (0, − 1) as the early flow is essentially inviscid away from the wall. The leading term in (6.4) represents the vorticity on the wall due to the thin shear layer induced for small times by this dipole and its image. The inviscid flow due to the pair of dipoles has stagnation points on the boundary at (± 1, 0), and so there is a separated flow commencing at $t = 0$. When the

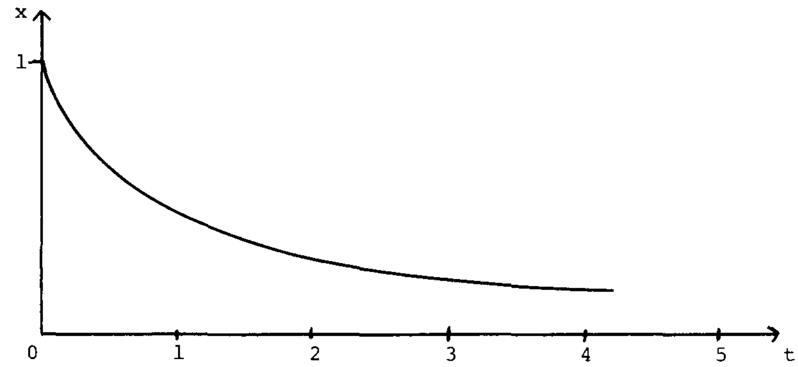

Fig. 4. Position of the point of separation $x = x_w(t)$ on the wall for the flow due to a stokeslet parallel to the wall.

higher-order terms (up to the $O(t^{3/2})$ term) in (6.4) are included, it is seen that the vorticity is zero on the wall at $x = \pm x_w(t)$ where

$$x_w = 1 - \tfrac{1}{2}(\pi t)^{1/2} + \tfrac{3}{8}\pi t + O(t^{3/2}),$$

so the point of separation begins to move in towards the origin; the vorticity is positive on the wall for $|x| > x_w$, and negative for $|x| < x_w$. Secondly, for large t, it can be seen for finite x that

$$\pi\omega(x, 0, t) = \frac{8x^2}{(1 + x^2)^2} - \frac{1}{t} + O(t^{-3/2});$$

the first term represents the steady-state value, which is zero at the origin, but otherwise positive for all $x \neq 0$. It follows that $x_w \simeq (8t)^{-1/2}$ for $t \gg 1$, and so $x_w \to 0$ as $t \to \infty$. The position of the point of separation $(\pm x_w, 0)$ on the wall is gained numerically from (6.3) on setting $\omega(x, 0, t) = 0$, and is sketched in Fig. 4. Thirdly, we see that

$$\omega(0, 0, t) = 4\pi^{-1}\left[-2\left\{t + \left(\frac{t}{\pi}\right)^{1/2}\right\}(1 - e^{-1/(4t)}) - 2t\,\mathrm{erfc}\left(\frac{1}{2t^{1/2}}\right)(e^{1/(4t)} - 1) + e^{-1/(4t)}\right],$$

which is negative for all $t > 0$. Consequently, it is clear that a separated region develops between the stokeslet and the wall immediately after the impulsive start to the motion, but that it decreases in size as t increases, ultimately collapsing at the origin as t tends to infinity; there is no indication of its existence for the steady flow.

We conclude by just presenting, for completeness, the solution for the stokeslet at (0, 1) when it is positioned in the positive y-direction. The vorticity $\omega(x, y, t)$ is given by

$$\pi\omega = -\frac{2x}{x^2 + (y - 1)^2}e^{-[x^2 + (y-1)^2]/(4t)}$$

$$- 16\left(\frac{t}{\pi}\right)^{1/2}\frac{x(y + 1)}{\{x^2 + (y + 1)^2\}^2}[e^{-y^2/(4t)} - e^{-(y+1)^2/(4t)}]$$

$$+\left[\frac{2x}{x^2+(y+1)^2}+\frac{4x(y+1)^2}{\{x^2+(y+1)^2\}^2}+\frac{8tx\{3(y+1)^2-x^2\}}{\{x^2+(y+1)^2\}^3}\right]$$

$$\times\ \mathrm{e}^{-[x^2+(y+1)^2]/(4t)}+\left[\frac{x}{x^2+(y-1)^2}-\frac{x}{x^2+(y+1)^2}\right.$$

$$\left.+\frac{4xy(y+1)}{\{x^2+(y+1)^2\}^2}-\frac{8tx\{3(y+1)^2-x^2\}}{\{x^2+(y+1)^2\}^3}\right]\mathrm{erfc}\left(\frac{y}{2t^{1/2}}\right)$$

$$+\left[-\frac{4x(y+1)^2}{\{x^2+(y+1)^2\}^2}+\frac{8tx\{3(y+1)^2-x^2\}}{\{x^2+(y+1)^2\}^3}\right]\mathrm{erfc}\left(\frac{y+1}{2t^{1/2}}\right)$$

$$+\ i\left[\frac{4x^2(y+1)}{\{x^2+(y+1)^2\}^2}-\frac{8t(y+1)\{(y+1)^2-3x^2\}}{\{x^2+(y+1)^2\}^3}\right]\mathrm{e}^{-[x^2+(y+1)^2]/(4t)}$$

$$\times\ \mathrm{erf}\left(\frac{ix}{2t^{1/2}}\right)+\left\{\left[-\frac{1}{2}\frac{x+i(y-1)}{x^2+(y-1)^2}-\frac{3}{2}\frac{x-i(y+1)}{x^2+(y+1)^2}\right.\right.$$

$$+\frac{2x(x^2+1+y)-i\{(y+2)(y+1)^2+x^2(3y+2)\}}{\{x^2+(y+1)^2\}^2}$$

$$\left.-\frac{4t[x\{3(y+1)^2-x^2\}-i(y+1)\{(y+1)^2-3x^2\}]}{\{x^2+(y+1)^2\}^3}\right]$$

$$\left.+\ [\text{complex conjugate}]\right\}.$$

The flow is symmetric about the line $x = 0$, and it can be shown that there is no separation of the flow along the wall at any time during the transition to the steady-state behaviour.

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